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PERIODIC AND ASYMPTOTIC ORBITS IN A FIVE BODY PROBLEM[1]

BY DANIEL BUCHANAN[2]

Abstract

Periodic and asymptotic orbits are obtained for the motion of an infinitesimal body subject to the Newtonian attraction of four finite bodies. The finite bodies are equal in mass and remain relatively fixed at the vertices of a square while they revolve about their common centre of gravity with uniform angular velocity. The infinitesimal body moves in the vicinity of the 13 libration points of the finite bodies. Asymptotic orbits of two dimensions and periodic orbits of two and of three dimensions are obtained.

1. Introduction

This paper deals with the determination of periodic and asymptotic orbits in the vicinity of the libration points in a restricted problem of five bodies. Four of the bodies are of equal finite masses and are assumed to be fixed at the vertices of a square that rotates uniformly about the centre of gravity of the four bodies. The fifth body is infinitesimal. It is attracted by the finite bodies according to the Newtonian law of the inverse square. Periodic orbits both in, and cutting, the plane of motion of the finite bodies and asymptotic orbits within the above plane of motion are obtained for the infinitesimal body moving in the vicinity of its points of libration.

2. The Libration Points

In a recent paper Hinrichsen (17) considered the libration or equilibrium points in the general problem of n bodies in which one body is infinitesimal and the other $n - 1$ bodies are of equal finite masses. The finite bodies are relatively fixed at the vertices of a regular $(n - 1)$-sided polygon and are rotating uniformly about their common centre of gravity.

The following axes and units were selected by Hinrichsen. The motion of the infinitesimal body is referred to a set of rectangular co-ordinate axes that rotate uniformly with the finite bodies and have the origin at their common centre of gravity. The x-axis is so chosen that, for n even, one vertex of the polygon lies on the negative half of the x-axis, and for n odd the vertices lie on both positive and negative sides of the x-axis. The co-

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ordinates of the finite bodies are denoted by $(x_i, y_i, 0)$, $(i = 1, ---, n-1)$, and those of the infinitesimal by (x, y, z). The unit of mass is chosen as the sum of the masses of the finite bodies and the unit of distance as the distance from any of the finite bodies to the centre of gravity of the other $n-2$ bodies. The unit of time is so taken that the angular velocity is unity.

With the axes and units thus chosen, the differential equations of motion for the infinitesimal body are

$$\left.\begin{aligned} \frac{d^2x}{dt^2} - 2\frac{dy}{dt} &= \Omega_x, \\ \frac{d^2y}{dt^2} + 2\frac{dx}{dt} &= \Omega_y, \\ \frac{d^2z}{dt^2} &= \Omega_z, \\ \Omega &= \frac{1}{2(n-1)} \sum_{i=1}^{n-1} \left(r_i^2 + \frac{2}{r_i}\right) \\ r_i^2 &= (x - x_i)^2 + (y - y_i)^2 + z^2, \quad i = 1, ---, n-1. \end{aligned}\right\} \tag{1}$$

These equations admit the Jacobi integral

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2 = 2\Omega - C,$$

where C is a constant of integration.

The libration points, as found by Hinrichsen, following the methods of Lagrange (18, Vol. 6, pp. 229-324; 20, pp. 309-318) and others, are the solutions of

$$\left.\begin{aligned} \Omega_x &= \sum_{i=1}^{n-1} \left(1 - \frac{1}{r_i^3}\right)(x - x_i) = 0, \\ \Omega_y &= \sum_{i=1}^{n-1} \left(1 - \frac{1}{r_i^3}\right)(y - y_i) = 0, \\ z &= 0. \end{aligned}\right\} \tag{2}$$

On multiplying the first and second of these equations by $(y - y_j)$ and $(x - x_j)$, respectively, and subtracting we obtain

$$\sum_{i=1}^{n-1} [(y_i - y_j)\, x - (x_i - x_j)\, y + x_i y_j - x_j y_i)] \left(1 - \frac{1}{r_i^3}\right) = 0, \quad j = 1, 2, ---, n-1. \tag{3}$$

These equations denote the various lines through the vertices of the $(n-1)$-sided polygon and also the circles of unit radius having their centres at the vertices of the polygon. These lines and circles divide the plane into various regions in certain of which the equations $\Omega_x = \Omega_y = 0$, $z = 0$ have solutions. For $n = 5$, the largest value of n considered in detail by Hinrichsen, the plane

is divided as in Fig. 1, the libration points being in the shaded regions. They are found to be as follows:

$$\left.\begin{array}{ll} (0, 0) & \\ (\pm\lambda_1, 0), \quad (0, \pm\lambda_1) & 1.25 < \lambda_1 < 1.26, \\ (\pm\lambda_2, \pm\lambda_2) & 0.46 < \lambda_2 < 0.47, \\ (\pm\lambda_3, \pm\lambda_3) & 0.62 < \lambda_3 < 0.63. \end{array}\right\} \tag{4}$$

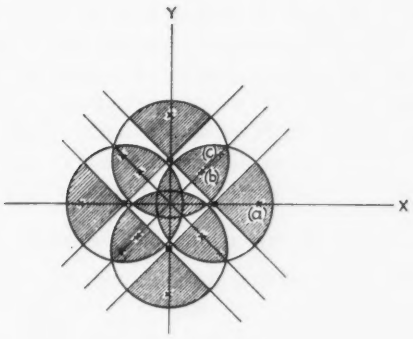

FIG. 1

In all there are 13 points, all points being within a distance 7/4 from the origin.

3. The Results Obtained

Owing to the symmetry of the problem, orbits only in the vicinity of the following are considered:

The origin,	(0, 0)
(*a*)	(1.255, 0),
(*b*)	(0.465, 0.465),
(*c*)	(0.625, 0.625).

The corresponding orbits near the nine other points can be obtained by interchanging or changing the signs of x and y and the co-ordinates of (*a*), (*b*), (*c*).

The method used in finding the periodic orbits follows somewhat closely that used by Moulton (21, Chap. V) in his treatment of an analogous three body problem. We use, however, the operator D of differential equations instead of the more laborious transformation to normal form used by Moulton.

Similar methods were used by Buck (16, Chap. IX), H. E. Buchanan (14, 15), and the author (1, Chap. 10, 2-6)* in treating various problems in periodic orbits.

**Other articles, chiefly in the Transactions of the Royal Society of Canada.*

The asymptotic orbits are found by the method used by the present author in certain other analogous problems (7-13)*.

Periodic orbits of two dimensions are obtained for motion near (*a*) and (*b*). These orbits correspond to Moulton's orbits of Class B (21, Chap. V). Asymptotic orbits of two dimensions are found near the four points, the origin, (*a*), (*b*), and (*c*). In these orbits the infinitesimal body approaches the equilibrium points as the time approaches infinity.

For motion outside of the *xy*-plane, periodic orbits for all four points are obtained. The orbit near the origin, however, is linear, being along the *z*-axis. These orbits correspond to Moulton's orbits of Class A.

Orbits of Moulton's category Class C were not considered. They depend upon the commensurability of two parameters, and their determination even when this commensurability is assumed is quite complex.

Asymptotic orbits of three dimensions similar to those obtained by the author in other cases are deferred for future treatment.

4. The Equations of Displacement

In order to discuss the motion in the vicinity of the equilibrium points we substitute in Equations (1)

$$x = \lambda + \epsilon u, \quad y = \mu + \epsilon v, \quad z = 0, \quad t = (1 + \delta)\tau, \tag{5}$$

where (λ, μ) are the co-ordinates of an equilibrium point; u, v are new dependent variables; τ is the new dependent variable; ϵ is a parameter denoting the scale factor of the orbits; and δ is a parameter vanishing with ϵ. On substituting Equations (5) in Equations (1) we obtain

$$\left.\begin{aligned} \epsilon D^2u - 2\epsilon(1+\delta)Dv &= \tfrac{1}{4}(1+\delta)^2 \sum_{i=1}^{4*} \left(1 - \frac{1}{\rho_i^3}\right)(\lambda - x_i + \epsilon u), \\ \epsilon D^2v + 2\epsilon(1+\delta)Du &= \tfrac{1}{4}(1+\delta)^2 \sum_{i=1}^{4*} \left(1 - \frac{1}{\rho_i^3}\right)(\mu - y_i + \epsilon v), \\ \rho_i^2 &= (\lambda - x_i + \epsilon u)^2 + (\mu - y_i + \epsilon v)^2, \qquad D \equiv \frac{d}{dt}. \end{aligned}\right\} \tag{6}$$

The asterisk in Σ^* denotes the omission of terms independent of ϵu or ϵv.

If we consider only the terms in Equations (6) that are linear in u and v we obtain, after dropping ϵ, the equations of variation,

$$\left.\begin{aligned} &(D^2 - d_1)u - (2D + d_2)v = 0, \\ &(2D - d_2)u + (D^2 - d_3)v = 0, \\ &d_1 = 1 - \tfrac{1}{4}\sum_{i=1}^{4}\left\{\frac{1}{A_i^3} - \frac{3}{A_i^5}(\lambda - x_i)^2\right\}, \\ &d_2 = \tfrac{3}{4}\sum_{i=1}^{4}\frac{1}{A_i^5}(\lambda - x_i)(\mu - y_i), \\ &d_3 = 1 - \tfrac{1}{4}\sum_{i=1}^{4}\left\{\frac{1}{A_i^3} - \frac{3}{A_i^5}(\mu - y_i)^2\right\}. \end{aligned}\right\} \tag{7}$$

*Other articles, chiefly in the Transactions of the Royal Society of Canada.

The values for the A's and d's for the equilibrium points considered are listed in Table I.

TABLE I

Libration point	(0, 0)	(1.255, 0)	(0.465, 0.465)	(0.625, 0.625)
$\frac{1}{A_1^3}$	$\left(\frac{4}{3}\right)^3$	7.765	6.162	3.862
$\frac{1}{A_2^3}$	$\left(\frac{4}{3}\right)^3$	.320	6.162	3.862
$\frac{1}{A_3^3}$	$\left(\frac{4}{3}\right)^3$	.124	.454	.290
$\frac{1}{A_4^3}$	$\left(\frac{4}{3}\right)^3$	.320	.454	.290
$\frac{1}{A_1^5}$	$\left(\frac{4}{3}\right)^5$	30.45	20.72	9.507
$\frac{1}{A_2^5}$	$\left(\frac{4}{3}\right)^5$	.150	20.72	9.507
$\frac{1}{A_3^5}$	$\left(\frac{4}{3}\right)^5$	.031	.268	127
$\frac{1}{A_4^5}$	$\left(\frac{4}{3}\right)^5$	.150	.268	127
d_1	2.185	5.140	2.654	2.038
d_2	0	0	−3.892	− .950
d_3	d_1	−1.006	d_1	d_1

The indicial equation of Equations (7) is

$$D^4 + (4 - d_1 - d_3)\, D^2 + d_1\, d_3 - d_2^2 = 0. \tag{8}$$

Its roots for the various equilibrium points are listed in Table II.

TABLE II

Libration point	Indicial equation	Roots
(0, 0)	$D^4 - 0.370\, D^2 + 4.774 = 0$	$\pm (1.09 + i)$
(1.255, 0)	$D^4 - 0.134\, D^2 - 5.171 = 0$	$\pm$ 1.53, $\pm$ 1.49i
(0.465, 0.465)	$D^4 - 1.308\, D^2 - 8.104 = 0$	$\pm$ 1.88, $\pm$1.50i
(0.625, 0.625)	$D^4 - 0.076\, D^2 + 3.251 = 0$	$\pm (0.96 \pm 0.94i)$

Now the oscillations near a libration point are stable and periodic for those roots of the corresponding indicial equations that are purely imaginary, and unstable and asymptotic for the roots that are real or complex. Hence there are asymptotic orbits near all the libration points but periodic orbits near (1.255, 0) and (0.465, 0.465) only.

5. Periodic Orbits in the Plane

The periodic orbits will be constructed first. As the construction is similar for both points (1.255, 0) and (0.465, 0.465), only the latter point will be considered in detail.

In this case the solutions of the equations of variation are

$$\left.\begin{aligned} u &= B_1 e^{\alpha\tau} + B_2 e^{-\alpha\tau} + B_3 \cos\beta\tau + B_4 \sin\beta\tau, \\ v &= \gamma_1 B_1 e^{\alpha\tau} + \gamma_2 B_2 e^{-\alpha\tau} + (0.79\,B_3 + 0.61\,B_4)\cos\beta\tau \\ &\qquad + (-0.61\,B_3 + 0.79\,B_4)\sin\beta\tau, \\ \alpha &= 1.88, \qquad \beta = 1.50, \\ \gamma_1 &= \frac{\alpha^2 - d_1}{2\alpha + d_2} = -8.31, \qquad \gamma_2 = -\frac{\alpha^2 - d_1}{2\alpha - d_2} = -0.12. \\ B_1, &\ \cdots, B_4, \text{ arbitrary constants.} \end{aligned}\right\} \tag{9}$$

The trigonometric rather than the exponential form is used for the periodic solutions in order to simplify the construction.

For sufficiently small numerical values of ϵ the terms in Equations (6) may be expanded in converging series. The expansion yields, after ϵ has been cancelled off,

$$\left.\begin{aligned} [D^2 - d_1(1+\delta)^2]u - [2D(1+\delta) + d_2(1+\delta)^2]v \\ \qquad = (1+\delta)^2 \sum_{i=1}^{\infty} \sum_{j,k=0}^{i} U_{jk}^{(i)}\, u^j v^k \epsilon^i, \\ [2D(1+\delta) - d_2(1+\delta)^2]u + [D^2 - d_1(1+\delta)^2]v \\ \qquad = (1+\delta)^2 \sum_{i=1}^{\infty} \sum_{j,k=0}^{i} V_{jk}^{(i)}\, u^j v^k \epsilon^i, \\ j + k = i + 1, \end{aligned}\right\} \tag{10}$$

where $U_{jk}^{(i)}$, $V_{jk}^{(i)}$ are constants and $U_{jk}^{(i)} = V_{ki}^{(i)}$.

So far as the computation has been carried out we have

$$\begin{aligned} U_{20}^{(1)} &= \sum_{i=1}^{4} \frac{3}{8}\left[\frac{3}{A_i^5}(\lambda - x_i) - \frac{5}{A_i^7}(\lambda - x_i)^3\right], \\ U_{11}^{(1)} &= \sum_{i=1}^{4} \frac{3}{4}\left[\frac{1}{A_i^5}(\mu - y_i) - \frac{5}{A_i^7}(\lambda - x_i)^2(\mu - y_i)\right], \\ U_{02}^{(1)} &= \sum_{i=1}^{4} \frac{3}{8}\left[\frac{1}{A_i^5}(\lambda - x_i) - \frac{5}{A_i^7}(\lambda - x_i)(\mu - y_i)^2\right], \\ V_{02}^{(1)} &= \sum_{i=1}^{4} \frac{3}{8}\left[\frac{3}{A_i^5}(\mu - y_i) - \frac{5}{A_i^7}(\mu - y_i)^3\right], \end{aligned}$$

$$V_{11}^{(1)} = \sum_{i=1}^{4} \frac{3}{4}\left[\frac{1}{A_i^5}(\lambda - x_i) - \frac{5}{A_i^7}(\lambda - x_i)(\mu - y_i)^2\right],$$

$$V_{20}^{(1)} = \sum_{i=1}^{4} \frac{3}{8}\left[\frac{1}{A_i^5}(\mu - y_i) - \frac{5}{A_i^7}(\lambda - x_i)^2(\mu - y_i)\right].$$

For the equilibrium points having $\lambda = \mu$

$$U_{20}^{(1)} = V_{02}^{(1)},\ U_{11}^{(1)} = V_{11}^{(1)},\ U_{02}^{(1)} = V_{20}^{(1)}.$$

If

$$\lambda = \mu = 0.465$$

then

$$U_{20}^{(1)} = V_{02}^{(1)} = -5.97,$$
$$U_{11}^{(1)} = V_{11}^{(1)} = 8.80,$$
$$U_{02}^{(1)} = V_{20}^{(1)} = 4.40.$$

Now let

$$u = \sum_{j=0}^{\infty} u_j \epsilon^j,\ v = \sum_{j=0}^{\infty} v_j \epsilon^j,\ \delta = \sum_{j=1}^{\infty} \delta_j \epsilon^j \tag{11}$$

be substituted in Equations (10) and let the resulting equations be cited as Equations (10′). On equating the various powers of ϵ in Equations (10′) we obtain sets of differential equations defining the u_j and v_j. It will be shown that the δ_j and the arbitrary constants at each step of the integration can be determined so that u_j and v_j will be periodic with the period $2\pi/\beta$ and will satisfy certain assigned initial conditions. Because of symmetry it will be supposed that $u = v = 1$ at $\tau = 0$. There is no loss of generality in choosing unity inasmuch as u and v carry the arbitrary factor ϵ in Equation (5). With these initial values we have, at $\tau = 0$,

$$u_0 = v_0 = 1, \quad u_j = v_j = 0, j > 0. \tag{12}$$

Step 1: Coefficients of ϵ^0 in Equations (10′)

The differential equations at this step are the same as the equations of variation. Their solutions are therefore the same as Equations (9). When the periodicity and initial conditions are imposed we obtain

$$\left.\begin{aligned} u_0 &= \cos\beta\tau + 0.34\sin\beta\tau, \\ v_0 &= \cos\beta\tau - 0.34\sin\beta\tau, \end{aligned}\right\} \tag{13}$$

Step 2: Coefficients of ϵ in Equations (10′)

At the second step the differential equations are

$$\left.\begin{aligned} (D^2 - d_1)u_1 - (2D + d_2)v_1 &= U^{(2)}, \\ (2D - d_2)u_1 + (D^2 - d_1)v_1 &= V^{(2)}, \end{aligned}\right\} \tag{14}$$

where

$$U^{(2)} = \delta_1(-3.50\cos\beta\tau + 1.45\sin\beta\tau) + 2.99 + 4.24\cos 2\beta\tau - 3.53\sin 2\beta\tau,$$

$$V^{(2)} = \delta_1(-3.50\cos\beta\tau - 1.45\sin\beta\tau) + 2.99 + 4.24\cos 2\beta\tau + 3.53\sin 2\beta\tau.$$

The complementary functions having the period $2\pi/\beta$ are

$$\left.\begin{aligned} u_1 &= B_3^{(1)} \cos \beta\tau + B_4^{(1)} \sin \beta\tau, \\ v_1 &= (0.79\ B_3^{(1)} + 0.61\ B_4^{(1)}) \cos \beta\tau \\ &\qquad + (-0.61\ B_3^{(1)} + 0.79\ B_4^{(1)}) \sin \beta\tau, \end{aligned}\right\} \tag{15}$$

where $B_3^{(1)}$ and $B_4^{(1)}$ are constants of integration.

The particular integrals, in symbolic form, are

$$\left.\begin{aligned} u_1 &= \frac{(D^2 - d_1)U^{(2)} + (2D + d_2)V^{(2)}}{\Delta}, \\ v_1 &= \frac{-(2D - d_2)U^{(2)} + (D^2 - d_1)V^{(2)}}{\Delta}, \\ \Delta &= D^4 + (4 - 2d_1)D^2 + d_1^2 - d_2^2. \end{aligned}\right\} \tag{16}$$

Since Δ vanishes for $D = \beta$ the terms in $\cos \beta\tau$ and $\sin \beta\tau$ in the numerators of Equations (16) must vanish. These terms are

$$\left.\begin{aligned} &\delta_1(D^2 - d_1)\{-3.50 \cos \beta\tau + 1.45 \sin \beta\tau\} + \\ &\qquad \delta_1(2D + d_2)\{-3.50 \cos \beta\tau - 1.45 \sin \beta\tau\} = 0 \\ \text{and} \\ &-\delta_1(2D - d_2)\{-3.50 \cos \beta\tau + 1.45 \sin \beta\tau\} + \\ &\qquad \delta_1(D^2 - d_1)\{-3.50 \cos \beta\tau - 1.45 \sin \beta\tau\} = 0. \end{aligned}\right\} \tag{17}$$

The determinant of the terms in the brackets { } vanishes for $D = \beta$ and hence one equation in Equations (17) is a consequence of the other. Selecting the first equation we must have vanishing coefficients for both $\cos \beta\tau$ and $\sin \beta\tau$.

Thus

$$\delta_1[3.50(\beta^2 + d_1 - d_2) - 1.45(2\beta)] = 0,$$

$$\delta_1[3.50(2\beta) - 1.45(\beta^2 + d_1 + d_2)] = 0.$$

Now the determinant of $3.50\ \delta_1$ and $-1.45\ \delta_1$ in these two equations vanishes and hence both equations are satisfied when either equation is. From the first equation we obtain $\delta_1 = 0$. With this value of δ_1 the particular integrals become

$$\left.\begin{aligned} u_1 &= 2.42 - 0.09 \cos 2\beta\tau + 0.06 \sin 2\beta\tau, \\ v_1 &= 2.42 - 0.09 \cos 2\beta\tau - 0.06 \sin 2\beta\tau. \end{aligned}\right\} \tag{18}$$

Combining these particular integrals with Complementary Functions (15) and selecting the constants of integration $B_3^{(1)}$ and $B_4^{(1)}$ so that $u_1(0) = v_1(0) = 0$, we obtain the desired solutions

$$\left.\begin{aligned} u_1 &= 2.42 - (2.33\cos\beta\tau + 0.80\sin\beta\tau) \\ &\qquad -0.09\cos 2\beta\tau + 0.06\sin 2\beta\tau, \\ v_1 &= 2.42 - (2.33\cos\beta\tau - 0.80\sin\beta\tau) \\ &\qquad -0.09\cos 2\beta\tau - 0.06\sin 2\beta\tau. \end{aligned}\right\} \tag{19}$$

Step 3: Coefficients of ϵ^2 in Equations (10′)

It is necessary to consider in detail one more step of the integration before proceeding to the general step. The differential equations at the third step are

$$\left.\begin{aligned} &(D^2 - d_1)u_2 - (2D + d_2)v_2 = U^{(3)}, \\ &(2D - d_2)u_2 + (D^2 - d_1)v_2 = V^{(3)}, \\ &U^{(3)} = U^{(3)}(\tau) = \delta_2(-3.50\cos\beta\tau + 1.45\sin\beta\tau) \\ &\qquad + \sum_{j=0}^{3}(C_j^{(3)}\cos j\beta\tau + S_j^{(3)}\sin j\beta\tau), \\ &V^{(3)} = U^{(3)}(-\tau). \end{aligned}\right\} \tag{20}$$

where $C_j^{(3)}$, $S_j^{(3)}$ are known constants. The terms containing δ_2 as a factor occur only as written. In order that the particular integrals of Equations (20) may be periodic we must have, as in Equations (17),

$$\left.\begin{aligned} &(D^2-d_1)\{(-3.50\,\delta_2+C_0^{(3)})\cos\beta\tau+(1.45\delta_2+S_0^{(3)})\sin\beta\tau\} \\ &+(2D+d_2)\{(-3.50\,\delta_2+C_0^{(3)})\cos\beta\tau-(1.45\delta_2+S_0^{(3)})\sin\beta\tau\} = 0, \\ &-(2D-d_2)\{(-3.50\,\delta_2+C_0^{(3)})\cos\beta\tau+(1.45\delta_2+S_0^{(3)})\sin\beta\tau\} \\ &+(D^2-d_1)\{(-3.50\delta_2+C_0^{(3)})\cos\beta\tau-(1.45\delta_2+S_0^{(3)})\sin\beta\tau\} = 0. \end{aligned}\right\} \tag{21}$$

As at the previous step the functional determinant of the terms in the brackets $\{\ \}$ vanishes for $D = \beta$. Hence one equation of Equations (21) is a consequence of the other. Selecting the first equation, we must have vanishing coefficients for both $\cos\beta\tau$ and $\sin\beta\tau$. Thus

$$\left.\begin{aligned} &(\beta^2 + d_1 - d_2)\{3.50\delta_2 - C_0^{(3)}\} - 2\beta\{1.45\delta_2 + S_0^{(3)}\} = 0, \\ &2\beta\{3.50\delta_2 - C_0^{(3)}\} - (\beta^2 + d_1 + d_2)\{1.45\delta_2 + S_0^{(3)}\} = 0. \end{aligned}\right\} \tag{22}$$

The determinant of the terms in the brackets $\{\ \}$ vanishes for $\beta = 1.50$ and hence one equation is a consequence of the other. Solving the first one for δ_2 we obtain

$$\delta_2 = \frac{8.80\,C_0^{(3)} + 3\,S_0^{(3)}}{26.45}.$$

With δ_2 thus determined the particular integrals will have the same form as $U^{(3)}$ and $V^{(3)}$ if the terms in $\cos \beta\tau$ and $\sin \beta\tau$ are omitted. Thus the complete solutions which are periodic and satisfy the initial conditions have the form

$$u_2 = u_2(\tau) = \sum_{j=0}^{3} (c_j^{(3)} \cos j\beta\tau + s_j^{(3)} \sin j\beta\tau)$$

$$v_2 = u_2(-\tau),$$

where $c_j^{(3)}$, $s_j^{(3)}$ are known constants and $\sum_{j=0}^{3} c_j^{(3)} = 0$.

The General Step

The remaining steps of the integration are similar to the preceding step. In solving for u_n and v_n it will be necessary and possible to determine δ_n so that the coefficients of $\cos \beta\tau$ and $\sin \beta\tau$ in the equations analogous to Equations (21) will vanish. The coefficients of δ_n in these equations are the same as those of δ_2 at the third step, and hence δ_n can be so chosen that the solutions will be periodic. When the initial and periodicity conditions are imposed, the solutions take the form

$$\left.\begin{aligned} u_n &= u_n(\tau) = \sum_{j=1}^{n+1} (c_j^{(n+1)} \cos j\beta\tau + s_j^{(n+1)} \sin j\beta\tau) \\ v_n &= u_n(-\tau), \qquad \delta_n = \text{constant}, \\ &\sum_{j=1}^{n+1} c_j^{(n+1)} = 0. \end{aligned}\right\} \tag{23}$$

On substituting Equations (23) in Equations (11) and the results in Equations (5) the final form of the periodic solutions near (*b*) is

$$\left.\begin{aligned} x &= x(\tau) = 0.465 + \sum_{n=1}^{\infty} \sum_{j=1}^{n} (c_j^{(n)} \cos j\beta\tau + s_j^{(n)} \sin j\beta\tau)\, \epsilon^n, \\ y &= x(-\tau), \qquad \delta = \sum_{n=2}^{\infty} \delta_n \epsilon^n. \end{aligned}\right\} \tag{24}$$

6. An Approximate Periodic Orbit

If we consider only the linear terms in ϵ in Periodic Solutions (24) we obtain

$$x - 0.465 = \epsilon(\cos \beta\tau + 0.34 \sin \beta\tau),$$

$$y - 0.465 = \epsilon(\cos \beta\tau - 0.34 \sin \beta\tau),$$

or

$$\frac{(x + y - 0.930)^2}{(2\epsilon)^2} + \frac{(x - y)^2}{(0.68\epsilon)^2} = 1.$$

This represents an ellipse with the centre at the equilibrium point and the axes rotated through 45°. The semi-axes are $\sqrt{2}\epsilon$ and $0.34\sqrt{2}\epsilon$.

If initial values other than $u(0) = v(0) = \epsilon$ were chosen, say $u(0) = \epsilon$, $v(0) = 0$, the approximate orbit would still be an ellipse oriented as in Fig. 2, the difference being that the initial projection in the latter case would not be from an apse.

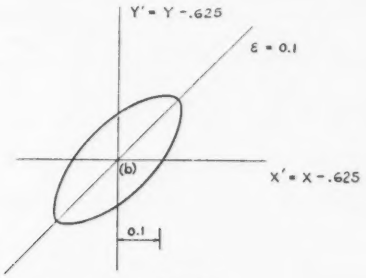

FIG. 2

7. Asymptotic Orbits Near (0.465, 0.465)

To construct the asymptotic orbits we begin with Equations (5) but do not change the independent variable t. The equations for the displacement from the equilibrium point (λ, μ) are the same as Equations (6) except that δ is neglected and D is considered as denoting d/dt. The generating solutions are, as in Equations (9),

$$u = B_1 e^{\alpha t} + B_2 e^{-\alpha t},$$
$$v = \gamma_1 B_1 e^{\alpha t} + \gamma_2 B_2 e^{-\alpha t},$$

where the periodic terms have been dropped.

Now substitute Equations (11), with $\delta = 0$, in Equations (10) and denote the resulting equations as Equations (10″). The solutions for u_j, v_j are then found by integrating Equations (10″) step by step and retaining only the terms in powers of $e^{-\alpha t}$. We choose the initial conditions $u(0) = 1$. The desired solutions at the first step are therefore

$$u_0 = e^{-\alpha t}, \quad v_0 = \gamma_2 e^{-\alpha t}.$$

Coefficients of ϵ in Equations (10″)

The differential equations at the second step are

$$(D^2 - d_1)u_1 - (2D + d_2)v_1 = U^{(2)} = B_2^{(2)} e^{-2\alpha t},$$
$$(2D - d_2)u_1 + (D^2 - d_1)v_1 = V^{(2)} = C_2^{(2)} e^{-2\alpha t},$$

where $B_2^{(2)}$ and $C_2^{(2)}$ are known constants. The complete solutions of these equations that satisfy the asymptotic and initial conditions are

$$u_1 = b_1^{(2)} e^{-\alpha t} + b_2^{(2)} e^{-2\alpha t},$$
$$v_1 = c_1^{(2)} e^{-\alpha t} + c_2^{(2)} e^{-2\alpha t},$$

where $b_1^{(2)}, - - -, c_2^{(2)}$ are known constants and $b_1^{(2)} + b_2^{(2)} = 0$.

The remaining steps of the integration are similar to this step. As there are no terms in $e^{\pm\alpha t}$ in the right members of the differential equations at any step, there will be no terms in $te^{\pm\alpha t}$ in the particular integrals. The solutions at the general step have the form

$$\left.\begin{aligned} u_j &= \sum_{k=1}^{j+1} b_k^{(j+1)} e^{-k\alpha t}, \\ v_j &= \sum_{k=1}^{j+1} c_k^{(j+1)} e^{-k\alpha t}, \end{aligned}\right\} \qquad (25)$$

where $b_k^{(j+1)}$, $c_k^{(j+1)}$ are known constants, and $\sum_{k=1}^{j+1} b_k^{(j+1)} = 0$.

On substituting Equations (25) in Equations (11) and the results in Equations (15) we obtain

$$x = 0.465 + \sum_{j=1}^{\infty} \sum_{k=1}^{j} b_k^{(j)} e^{-k\alpha t} \epsilon^j,$$

$$y = 0.465 + \sum_{j=1}^{\infty} \sum_{k=1}^{j} c_k^{(j)} e^{-k\alpha t} \epsilon^j,$$

where $b_k^{(j)}$, $c_k^{(j)}$ are known constants, and $\sum_{k=1}^{j} b_k^{(j)} = 0$.

Considering only the linear terms in ϵ we find an approximate orbit to be

$$x = 0.465 + e^{-\alpha t} \epsilon,$$

$$y = 0.465 - 0.12\, e^{-\alpha t} \epsilon.$$

This represents a straight line through the equilibrium point having the slope -0.12.

8. Orbits Near (1.255, 0)

Only the results for the orbits near the equilibrium point (1.255, 0) will be listed.

The Periodic Orbit

The initial conditions are so chosen that the infinitesimal is initially projected from the x-axis and at a distance ϵ from the equilibrium point. The solutions are

$$x - 1.255 = (\cos \beta\tau)\epsilon + \sum_{j=2}^{\infty} \sum_{k=1}^{j} (c_k^{(j)} \cos k\beta\tau)\epsilon^j,$$

$$y = -(4.93 \sin \beta\tau)\epsilon + \sum_{j=2}^{\infty} \sum_{k=1}^{j} (s_k^{(j)} \sin k\beta\tau)\epsilon^j,$$

$$\delta = \sum_{j=1}^{\infty} \delta_j \epsilon^j, \qquad \beta = 1.49,$$

where $c_k^{(j)}$, $s_k^{(j)}$ are constants. The approximate shape of the orbit is found in Fig. 3. It is an ellipse with the centre at the equilibrium point and the axes as indicated. The semi-axes are $4.93\ \epsilon$ and ϵ.

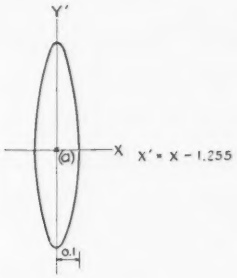

FIG. 3

The Asymptotic Orbit

The equations for the asymptotic orbit are

$$x - 1.255 = e^{-\alpha t}\epsilon + \sum_{i=2}^{\infty}\sum_{k=1}^{i} b_k^{(i)} e^{-k\alpha t}\epsilon^i,$$

$$y = 0.92\, e^{-\alpha t}\epsilon + \sum_{i=2}^{\infty}\sum_{k=1}^{i} c_k^{(i)} e^{-k\alpha t}\epsilon^i,$$

$$\alpha = 1.53.$$

The initial conditions are chosen so that the projection on the x-axis of the initial displacement from the equilibrium point is ϵ. The approximate orbit is a line through the equilibrium point and having the slope 0.92.

9. Asymptotic Orbits Near (0.625, 0.625)

As there are no purely imaginary roots of Indicial Equation (8) for the equilibrium point (0.625, 0.625) there are no periodic but only asymptotic orbits near this point. Since the roots are complex, and not real, the construction of the asymptotic orbits differs somewhat from that used for the asymptotic orbits near (0.425, 0.425).

As before we begin with Equations (10″), i.e., with Equations (10′) having $\delta = 0$, $D = d/dt$. We then equate the coefficients of the various powers of ϵ in Equations (10″) and, by a proper choice of the constants of integration, retain only those terms that are of the form $e^{-kt}P(t)$, where k is real and positive and $P(t)$ is a periodic function of t. As there are two undetermined constants at each step of the integration, we shall suppose $u = v = 1$ at $t = 0$. Hence

$$u_0(0) = v_0(0) = 1, \qquad u_j(0) = v_j(0) = 0, j > 0.$$

Step 1: Coefficients of ϵ^0 in Equations (10″)

The differential equations at the first step are the same as the equations of variation, the general solutions of which are

$$\left.\begin{aligned} u_0 &= b_1^{(1)} e^{(\alpha+i\beta)t} + b_2^{(1)} e^{(\alpha-i\beta)t} + b_3^{(1)} e^{-(\alpha+i\beta)t} + b_4^{(1)} e^{-(\alpha-i\beta)t}, \\ v_0 &= \gamma_1 b_1^{(1)} e^{(\alpha+i\beta)t} + \gamma_2 b_2^{(1)} e^{(\alpha-i\beta)t} + \gamma_3 b_3^{(1)} e^{-(\alpha+i\beta)t} + \gamma_4 b_4^{(1)} e^{-(\alpha-i\beta)t}, \\ \gamma_1 &= 0.32 + 1.23\,i, \qquad \gamma_3 = 0.20 - 0.76\,i, \\ \gamma_2 &= 0.32 - 1.23\,i, \qquad \gamma_4 = 0.20 + 0.76\,i. \end{aligned}\right\} \tag{26}$$

If the real part of the exponential is positive, as in $e^{(\alpha\pm i\beta)t}$, such terms do not approach zero as t approaches infinity and must therefore be rejected by putting $b_1^{(1)} = b_2^{(1)} = 0$. The desired solutions that satisfy the initial condition $u_0(0) = v_0(0) = 1$ are

$$\left.\begin{aligned} u_0 &= e^{-\alpha t}(\cos \beta t + 1.05 \sin \beta t), \\ v_0 &= e^{-\alpha t}(\cos \beta t - 0.55 \sin \beta t). \end{aligned}\right\} \tag{27}$$

Step 2: Coefficients of ϵ in Equations (10″)

The differential equations at this step are

$$\left.\begin{aligned} (D^2 - d_1)u_1 - (2D + d_2)v_1 &= U^{(2)}, \\ (2D - d_2)u_1 + (D^2 - d_1)v_1 &= V^{(2)}, \\ U^{(2)} &= e^{-2\alpha t}\,[E_0^{(2)} + E_2^{(2)} \cos 2\beta t + F_2^{(2)} \sin 2\beta t], \\ V^{(2)} &= e^{-2\alpha t}\,[G_0^{(2)} + G_2^{(2)} \cos 2\beta t + H_2^{(2)} \sin 2\beta t], \end{aligned}\right\} \tag{28}$$

where $E_0^{(2)}, - - -, H_2^{(2)}$ are real constants.

The complementary functions having the asymptotic property are

$$\begin{aligned} u_2 &= b_3^{(2)} e^{-(\alpha+i\beta)t} + b_4^{(2)} e^{-(\alpha-i\beta)t}, \\ v_2 &= \gamma_3 b_3^{(2)} e^{-(\alpha+i\beta)t} + \gamma_4 b_4^{(2)} e^{-(\alpha-i\beta)t}, \end{aligned}$$

where $b_3^{(2)}$, $b_4^{(2)}$ are arbitrary constants.

The particular integrals, in symbolic form, are

$$\left.\begin{aligned} u_1 &= \frac{(D^2 - d_1)\,U^{(2)} + (2D + d_2)\,V^{(2)}}{\Delta}, \\ v_1 &= \frac{-(2D - d_2)\,U^{(2)} + (D^2 - d_1)\,V^{(2)}}{\Delta}, \end{aligned}\right\} \tag{29}$$

where Δ is the determinant in Equation (8). Now terms in t explicitly can arise in Equations (29) only if terms in $e^{\pm(\alpha\pm i\beta)t}$ occur in $U^{(2)}$ and $V^{(2)}$, and no such terms appear in $U^{(2)}$ and $V^{(2)}$. To obtain the particular integrals we make use of the well known property

$$\begin{aligned} f(D)e^{\lambda t} \begin{matrix}\cos \\ \sin\end{matrix} \nu t &= e^{\lambda t} f(D + \lambda) \begin{matrix}\cos \\ \sin\end{matrix} \nu t, \\ \frac{1}{f(D)} e^{\lambda t} \begin{matrix}\cos \\ \sin\end{matrix} \nu t &= e^{\lambda t} \frac{1}{f(D + \lambda)} \begin{matrix}\cos \\ \sin\end{matrix} \nu t, \end{aligned}$$

where $f(D)$ is a polynomial in the operator D. We therefore find the particular integrals for u_1 and v_1 to have the same form as $U^{(2)}$ or $V^{(2)}$. Then the complete solutions that satisfy the asymptotic and initial conditions are

$$u_1 = e^{-\alpha t}[e_{11}^{(2)} \cos \beta t + f_{11}^{(2)} \sin \beta t] + e^{-2\alpha t}[e_{20}^{(2)} + e_{22}^{(2)} \cos 2\beta t + f_{22}^{(2)} \sin 2\beta t],$$

$$v_1 = e^{-\alpha t}[g_{11}^{(2)} \cos \beta t + h_{11}^{(2)} \sin \beta t] + e^{-2\alpha t}[g_{20}^{(2)} + g_{22}^{(2)} \cos 2\beta t + h_{22}^{(2)} \sin 2\beta t],$$

where $e_{11}^{(2)}, ---, h_{22}^{(2)}$ are real constants and

$$e_{11}^{(2)} + e_{20}^{(2)} + e_{22}^{(2)} = 0, \quad g_{11}^{(2)} + g_{20}^{(2)} + g_{22}^{(2)} = 0.$$

The General Step

The remaining steps of the integration are similar to Step 2. No terms in $te^{-\alpha t}$ occur in the particular integrals. The solutions for u_n and v_n have the form

$$\left.\begin{aligned} u_n &= e^{-\alpha t}\left[\begin{matrix}\cos\\ \sin\end{matrix} \beta t\right] + e^{-2\alpha t}\left[\text{const.}, \begin{matrix}\cos\\ \sin\end{matrix} 2\beta t\right] + ---- \\ &\quad + e^{-(n+1)\alpha t}\left[\begin{matrix}\cos\\ \sin\end{matrix} \beta t, \begin{matrix}\cos\\ \sin\end{matrix} 3\beta t, ---, \begin{matrix}\cos\\ \sin\end{matrix} (n+1)\beta t\right] \text{ if } n \text{ is even,} \\ &\quad \text{or } e^{-(n+1)\alpha t}\left[\text{const.}, \begin{matrix}\cos\\ \sin\end{matrix} 2\beta t, ---, \begin{matrix}\cos\\ \sin\end{matrix} (n+1)\beta t\right] \text{ if } n \text{ is odd,} \\ v_n &= \text{similar to } u_n. \end{aligned}\right\} \tag{30}$$

The expressions in the brackets [ ] denote sums of cosines and sines of the multiples of βt indicated. The constants of integration are so chosen that $u_n(0) = v_n(0) = 0$.

The final form of the solutions is

$$x = 0.625 + \sum_{n=1}^{\infty} u_{n-1}\epsilon^n, \qquad y = 0.625 + \sum_{n=1}^{\infty} v_{n-1}\epsilon^n.$$

If we consider only the linear terms in ϵ, we find an approximate orbit to be

$$x = 0.625 + e^{-\alpha t}(\cos \beta t - 0.55 \sin \beta t)\epsilon,$$

$$y = 0.625 + e^{-\alpha t}(\cos \beta t + 1.05 \sin \beta t)\epsilon,$$

$$\alpha = 0.96, \quad \beta = 0.94.$$

Table III gives the corresponding values for x and y for the various values of t listed.

TABLE III

t	$x - 0.625$	$y - 0.625$	t	$x - 0.625$	$y - 0.625$
0	1	1	1.2	−.022	.435
.1	.860	.991	1.4	−.079	.328
.2	.725	.973	1.6	−.103	.238
.3	.607	.930	1.8	−.118	.164
.4	.493	.900	2	−.122	.102
.5	.397	.848	3	−.063	−.0347
.6	.312	.787	4	−.0107	−.0307
.7	.226	.732	5	+.0045	−.0088
.8	.166	.672	6	+.0036	+.0005
.9	.104	.610	7	+.00095	+.0015
1	.056	.551	8	−.000088	+.0006

The graph of the approximate orbit is given in Fig. 4.

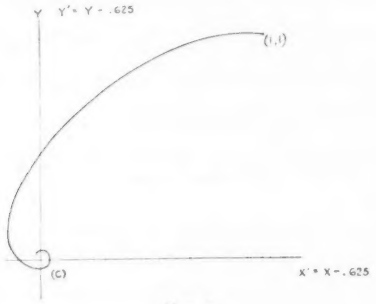

FIG. 4

10. Orbit in the Vicinity of the Origin

As for the equilibrium point (0.625, 0.625) only one type of orbit, viz., the asymptotic, exists in the vicinity of the origin. The solutions for this orbit are

$$x = \epsilon e^{-\alpha t}(\cos \beta t + \tfrac{1}{2} \sin \beta t) + \epsilon^2 \left[e^{-\alpha t} \begin{Bmatrix} \cos \\ \sin \end{Bmatrix} \beta t \right\} + e^{-2\alpha t} \left\{ \text{const.}, \begin{matrix} \cos \\ \sin \end{matrix} 2\beta t \right\} - - - \Bigg]$$

$$y = \epsilon e^{-\alpha t}(\cos \beta t - \tfrac{1}{2} \sin \beta t) + \epsilon^2 \left[e^{-\alpha t} \begin{Bmatrix} \cos \\ \sin \end{Bmatrix} \beta t \right\} + e^{-2\alpha t} \left\{ \text{const.}, \begin{matrix} \cos \\ \sin \end{matrix} 2\beta t \right\} - - - \Bigg]$$

$$\alpha = 1.09, \qquad \beta = 1.00.$$

The higher degree terms are similar to those in Equations (30). Table IV contains the values of x and y for various values of t for the approximate orbit obtained by taking only the linear terms in ϵ. Fig. 5 represents the orbit thus obtained.

TABLE IV

t	x	y	t	x	y
0	1	1	1.2	.23	−.028
.1	.94	.85	1.4	.15	−.071
.2	.87	.71	1.6	.083	−.093
.3	.79	.58	1.8	.037	−.101
.4	.78	.47	2	.0044	−.099
.5	.65	.37	3	−.0354	−.0408
.6	.58	.28	4	−.0134	−.0036
.7	.51	.21	5	−.00086	+.0033
.8	.44	.14	6	+.00121	+.0016
.9	.38	.086	7	+.00054	+.000212
1	.32	.040	8	+.000058	−.000107

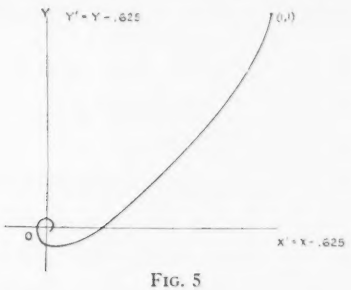

FIG. 5

11. Three Dimensional Periodic Orbits

We shall now consider certain three-dimensional periodic orbits in the vicinity of the equilibrium points.

To discuss the motion near a libration point we make in Equations (1) the substitutions

$$x = \lambda + \epsilon u, \quad y = \mu + \epsilon v, \quad z = \epsilon w, \quad t = (1 + \delta)\tau. \tag{31}$$

The resulting differential equations are the same as Equations (6) with the terms in w added.

They are

$$\begin{aligned}
\epsilon D^2 u - 2\epsilon(1 + \delta)Dv &= \tfrac{1}{4}(1 + \delta)^2 \sum_{i=1}^{4}{}^{*}\left(1 - \frac{1}{\rho_i^3}\right)(\lambda - x_i + \epsilon u), \\
\epsilon D^2 v + 2\epsilon(1 + \delta)Du &= \tfrac{1}{4}(1 + \delta)^2 \sum_{i=1}^{4}{}^{*}\left(1 - \frac{1}{\rho_i^3}\right)(u - y_i + \epsilon v), \qquad (32) \\
\epsilon D^2 w &= \tfrac{1}{4}(1 + \delta)^2 \sum_{i=1}^{4}\left(1 - \frac{1}{\rho_i^3}\right)(\epsilon w), \\
\rho_i^2 &= (\lambda - x_i + \epsilon u)^2 + (\mu - y_i + \epsilon v)^2 + \epsilon^2 w^2.
\end{aligned}$$

The equations of variation are

$$\begin{aligned}
(D^2 - d_1)u - (2D + d_2)v &= 0, \\
(2D - d_2)u + (D^2 - d_3)v &= 0, \qquad (33) \\
(D^2 + \gamma^2)w &= 0,
\end{aligned}$$

where d_1, d_2, d_3 have the same values as in Equations (7) and

$$\gamma^2 = -\tfrac{1}{4}\sum_{i=1}^{4}\left(1 - \frac{1}{A_i^3}\right).$$

The values of γ for the equilibrium points are as follows:

		(a)	(b)	(c)
	(0, 0)	(1.255, 0)	(0.465, 0.465)	(0.625, 0.625)
γ	1.17	1.06	1.52	1.04

The solutions of Equations (33) are

$$\left.\begin{aligned} &u \text{ and } v \text{ as in (9)} \\ &w = C_1 \cos \gamma\tau + C_2 \sin \gamma\tau, \end{aligned}\right\} \tag{34}$$

where C_1 and C_2 are constants of integration.

In order to obtain a three-dimensional orbit we must take as the generating solutions either

$$\left.\begin{aligned} &u = v = 0, \\ &w = C_1 \cos \gamma\tau + C_2 \sin \gamma\tau, \\ &\qquad \text{period} = 2\pi/\gamma, \\ \text{or} \\ &u = B_3 \cos \beta\tau + B_4 \sin \beta\tau, \\ &v = (0.79\, B_3 + 0.61\, B_4) \cos \beta\tau + \\ &\qquad (-0.61\, B_3 + 0.79\, B_4) \sin \beta\tau, \\ &w = C_1 \cos \gamma\tau + C_2 \sin \gamma\tau, \\ &\qquad \text{period} = n_1(2\pi/\beta) = n_2(2\pi/\gamma), \end{aligned}\right\} \tag{35}$$

where n_1, n_2 are relatively prime integers.

In the latter case, β and γ must be commensurable. In this latter case the orbits correspond to the orbits of Class C in Moulton's work. It is doubtful whether β and γ are commensurable, and consequently the construction of these orbits will not be considered.

Expanding Equations (32) as power series in ϵ and cancelling off ϵ, we obtain the equations corresponding to Equations (10). They are

$$\left.\begin{aligned} &[D^2 - d_1(1+\delta)^2]u - [2D(1+\delta) + d_2(1+\delta)^2]v \\ &\quad = \text{terms as in Equations (10)} + \tfrac{3}{8}\epsilon(1+\delta)^2 \sum_{i=1}^{4} (\lambda - x_i) \frac{1}{A_i^5} w^2 \\ &\qquad + \text{higher degree terms in } w^2, \\ &[2D(1+\delta) - d_2(1+\delta)^2]u + [D^2 - d_1(1+\delta)^2]v \\ &\quad = \text{terms as in Equations (10)} + \tfrac{3}{8}\epsilon(1+\delta)^2 \sum_{i=1}^{4} (\mu - y_i) \frac{1}{A_i^5} w^2 \\ &\qquad + \text{higher degree terms in } w^2, \\ &[D^2 + (1+\delta)^2\gamma^2]w = \tfrac{3}{4}(1+\delta)^2\epsilon \sum_{i=1}^{4} \left[\frac{1}{A_i^5}\{(\lambda - x_i)u + (\mu - y_i)v\}\right] w \\ &\qquad + \tfrac{3}{8}\epsilon^2 \sum_{i=1}^{4} \frac{1}{A_i^5} w^3 \\ &\qquad + \text{higher degree terms in } u, v, w. \end{aligned}\right\} \tag{36}$$

To guide the integration of Equations (36) as power series in ϵ we substitute

$$u = \sum_{j=0}^{\infty} u_j \epsilon^j, \quad v = \sum_{j=0}^{\infty} v_j \epsilon^j, \quad w = \sum_{j=0}^{\infty} w_j \epsilon^j, \quad \delta = \sum_{j=1}^{\infty} \delta_j \epsilon^j, \tag{37}$$

in Equations (36) and equate the coefficients of ϵ in the resulting equations, cited as Equations (36′).

Step 1: Coefficients of ϵ^0 in Equations (36′)

At the first step, differential equations that are the same as the equations of variation are obtained. The periodic generating solutions are

$$\begin{aligned} u_0 &= B_3^{(0)} \cos \beta\tau + B_4^{(0)} \sin \beta\tau, \\ v_0 &= (0.79\, B_3^{(0)} + 0.61\, B_4^{(0)}) \cos \beta\tau + (-0.61\, B_3^{(0)} + 0.79\, B_4^{(0)}) \sin \beta\tau, \\ &\qquad \text{period} = 2\pi/\beta, \\ w_0 &= C_1^{(1)} \cos \gamma\tau + C_2^{(1)} \sin \gamma\tau, \\ &\qquad \text{period} = 2\pi/\gamma. \end{aligned} \tag{38}$$

If we select the orbits having the period $2\pi/\beta$ we must put $w_0 = 0$ and this leads to the two-dimensional orbits already constructed. The possibility that β and γ are commensurable is disregarded, and hence the only new orbits which will arise are those having the period $2\pi/\gamma$. The solutions at the first step are therefore

$$\begin{aligned} u_0 &= v_0 = 0, \\ w_0 &= C_1^{(1)} \cos \gamma\tau + C_2^{(1)} \sin \gamma\tau. \end{aligned}$$

For initial conditions let it be supposed that $w(0) = 0$, $Dw(0) = \gamma$. Hence

$$w_j(0) = 0, \ j = 0, 1, 2, \cdots, \ Dw_0(0) = \gamma, \ Dw_j(0) = 0, \ j > 0. \tag{39}$$

The desired solutions at Step 1 are therefore

$$u_0 = v_0 = 0, \quad w_0 = \sin \gamma\tau.$$

Step 2: Coefficients of ϵ in Equations (36′)

At the second step we have

$$\left.\begin{aligned} (D^2 - d_1)u_1 - (2D + d_2)v_1 &= U_1 = \tfrac{3}{8} \sum_{i=1}^{4} (\lambda - x_i) \frac{w_0^2}{A_i^5} \\ (2D - d_2)u_1 + (D^2 - d_1)v_1 &= V_1 = \tfrac{3}{8} \sum_{i=1}^{4} (\mu - y_i) \frac{w_0^2}{A_i^5}, \\ (D^2 + \gamma^2)w_1 &= -2\delta_1 w_0. \end{aligned}\right\} \tag{40}$$

Beginning with Equation (40, 3), the last of Equations (40), we must have $\delta_1 = 0$ if w_1 is to be periodic. Further w_1 must likewise be zero to satisfy Equations (39). The complementary functions of Equations (40, 1 and 2) must be annulled to satisfy the periodicity conditions. Hence the solutions

at this step are the particular integrals of Equations (40, 1 and 2). The results for the different equilibrium points are listed in Table V.

TABLE V

	$(0, \mathring{0})$	(*a*) (1.255, 0)	(*b*) (0.465, 0.465)	(*c*) (0.625, 0.625)
U_1	0	$2.97\ (1 - \cos 2\gamma\tau)$	$0.78\ (1 - \cos 2\gamma\tau)$	$0.96\ (1 - \cos 2\gamma\tau)$
V_1	0	0	$0.78\ (1 - \cos 2\gamma\tau)$	$0.96\ (1 - \cos 2\gamma\tau)$
γ	—	1.06	1.52	1.04
$u_1(t)$	0	$-0.58 + 0.52 \cos 2\gamma\tau$	$0.63 + 0.14 \cos 2\gamma\tau + 0.05 \sin 2\gamma\tau$	$-0.88 + 0.32 \cos 2\gamma\tau + 0.18 \sin 2\gamma\tau$
v_1	0	$-0.80 \sin 2\gamma\tau$	$u_1(-\tau)$	$u_1(-\tau)$
δ_1	0	0	0	0
w_1	0	0	0	0

Step 3: Coefficients of ϵ^2 in Equations (36′)

Here the differential equations are

$$(D^2 - d_1)u_2 - (2D + d_2)v_2 = U_2 = 0,$$

$$(2D - d_2)u_2 + (D^2 - d_1)v_2 = V_2 = 0,$$

$$(D^2 + \gamma^2)w_2 = W_2 = -2\delta_2 w_0 + \tfrac{3}{4} \sum_{i=1}^{4} \frac{1}{A_i^5} \{(\lambda - x_i)u_1 + (\mu - y_i)v_i\} w_0 + \tfrac{3}{8} \sum_{i=1}^{4} \frac{1}{A_i^5} w_0^3.$$

Obviously $u_2 = v_2 = 0$.

In Table VI are listed the values of W_2, δ_2, w_2 for the four equilibrium points.

TABLE VI

	(0, 0)	(*a*) (1.255, 0)	(*b*) (0.465, 0.465)	(*c*) (0.625, 0.625)
W_2	$\left(-2\delta_2 + \frac{128}{27}\right) \sin \gamma\tau$	$(-2\delta_2 - 2.13) \sin \gamma\tau$	$(-2\delta_2 + 15.32) \sin \gamma\tau$	$(-2\delta_2 - 2.39) \sin \gamma\tau$
	$-\frac{128}{81} \sin 3\gamma\tau$	$+1.03 \sin 3\gamma\tau$	$-3.50 \sin 3\gamma\tau$	$-0.60 \sin 3\gamma\tau$
δ_2	64/27	−1.06	7.66	−1.20
w_2	$-0.42 \sin \gamma\tau$	$0.33 \sin \gamma\tau$	$-0.57 \sin \gamma\tau$	$-0.21 \sin \gamma\tau$
	$+0.14 \sin 3\gamma\tau$	$-0.11 \sin 3\gamma\tau$	$+0.19 \sin 3\gamma\tau$	$+0.07 \sin 3\gamma\tau$

The remaining integrations can be made as in the preceding steps.

Near the origin the u's and v's and the odd w's are all zero. The terms in w_{2j} consist of sums of sines of odd multiples of $\gamma\tau$, the highest multiple being $(2j + 1)\gamma\tau$.

In the vicinity of the other three libration points considered, the u's and v's with even subscripts and the w's and δ's with odd subscripts are all zero.

Libration point (a) (1.255, 0)

$$u_{2j+1} = \sum_{k=0}^{j+1} C_{a,2k}^{(2j+2)} \cos 2k\gamma\tau,$$

$$v_{2j+1} = \sum_{k=1}^{j+1} S_{a,2k}^{(2j+2)} \sin 2k\gamma\tau,$$

$$w_{2j} = \sum_{k=0}^{j} S_{a,2k+1}^{(2j+1)} \sin (2k+1)\gamma\tau.$$

Libration point (b) (0.465, 0.465)

$$\left.\begin{aligned} u_{2j+1}(\tau) &= \sum_{k=0}^{j+1} [C_{b,2k}^{(2j+2)} \cos 2k\gamma\tau + S_{b,2k}^{(2j+2)} \sin 2k\gamma\tau] \\ v_{2j+1}(\tau) &= u_{2j+1}(-\tau), \\ w_{2j} &= \sum_{k=0}^{j} S_{b,2k+1}^{(2j+1)} \sin (2k+1)\gamma\tau, \end{aligned}\right\} \tag{41}$$

where the C's and S's are constants.

Libration point (c) (0.625, 0.625)

The solutions are similar to Equations (41).

On substituting these results in Equations (38) and Equations (31) we obtain the following:

Libration point (0, 0)

$$x = y = 0, \quad z = \sum_{j=0}^{\infty} \sum_{k=0}^{j} [S_{0,2k+1}^{(2j+1)} \sin (2k+1)\gamma\tau]\epsilon^{2j+1},$$

Libration point (a) (1.255, 0)

$$x = 1.255 + \sum_{j=0}^{\infty} \sum_{k=0}^{j+1} [C_{a,2k}^{(2j+2)} \cos 2k\gamma\tau]\epsilon^{2j+2},$$

$$y = 1.255 + \sum_{j=0}^{\infty} \sum_{k=1}^{j+1} [S_{a,2k}^{(2j+2)} \sin 2k\gamma\tau]\epsilon^{2j+2},$$

Libration point (b) (0.465, 0.465)

$$x = x(\tau) = 0.465 + \sum_{j=0}^{\infty} \sum_{k=0}^{j+1} [C_{b,2k}^{(2j+2)} \cos 2k\gamma\tau + S_{b,2k}^{(2j+2)} \sin 2k\gamma\tau]\epsilon^{2j+2},$$
$$y = x(-\tau).$$

Libration point (c) (0.625, 0.625)

$$x = x(\tau) = 0.625 + \sum_{j=0}^{\infty} \sum_{k=0}^{j+1} [C_{c,2k}^{(2j+2)} \cos 2k\gamma\tau + S_{c,2k}^{(2j+2)} \sin 2k\gamma\tau]\epsilon^{2j+2}.$$

12. Approximate Orbits

Libration point (0, 0)

$$x = y = 0, \qquad z = (\sin \gamma\tau)\,\epsilon - 0.56\,\epsilon^3 \sin^3 \gamma\tau,$$

$$\gamma = 1.17, \qquad \delta = \frac{64}{27}\,\epsilon^2, \qquad \text{period} = 2\pi/\gamma.$$

This orbit lies wholly in the z-axis.

Libration point (a) (1.255, 0)

$$x = 1.255 - (0.58 - 0.66 \cos 2\gamma\tau)\,\epsilon^2,$$

$$y = \quad - (0.80 \sin 2\gamma\tau)\,\epsilon^2,$$

$$z = (\sin \gamma\tau)\,\epsilon + 0.44\,\epsilon^3 \sin^3 \gamma\tau,$$

$$\gamma = 1.06, \quad \delta = -1.06\,\epsilon^2, \quad \text{period} = 2\pi/\gamma.$$

The projections of the orbit on the planes of reference are as follows:

xy-plane

$$\frac{(x - 1.255 + 0.58\,\epsilon^2)^2}{(0.66\,\epsilon^2)^2} + \frac{y^2}{(0.8\,\epsilon^2)^2} = 1.$$

This is an ellipse with semi-axes, $0.66\,\epsilon^2$ and $0.8\,\epsilon^2$. The major axis is parallel to the y-axis. The graph is given in Fig. 6.

xz-plane

$$x - 1.255 - 0.08\,\epsilon^2 = -1.32\,z^2.$$

The orbit is a parabola, as in Fig. 7.

yz-plane

$$y = -1.60\,z\,\sqrt{\epsilon^2 - z^2}.$$

The orbit is a double loop, as in Fig. 8.

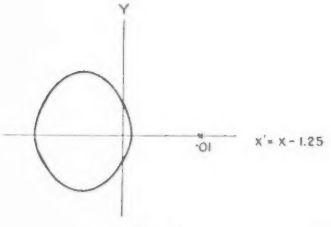

FIG. 6

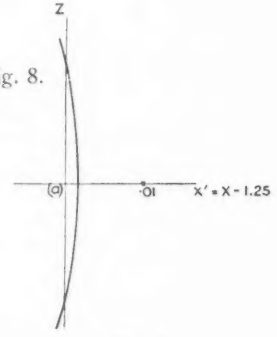

FIG. 7

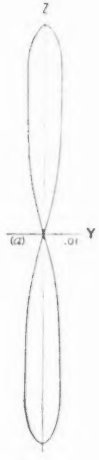

FIG. 8

The composite orbit is shown in Fig. 9 but no attempt has been made to draw the diagram to an accurate scale.

The spatial curve is a figure-8 bent around a tangential elliptic cylinder. The axis of the cylinder is vertical and the major axis of its elliptical cross-section is parallel to the y-axis. The tangential cylinder encloses Point (*a*).

Libration point (b) (0.465, 0.465)

$$x = x(\tau) = 0.465 + \epsilon^2(0.63 + 0.14\cos 2\gamma\tau + 0.05\sin 2\gamma\tau),$$
$$y = x(-\tau),$$
$$z = (\sin\gamma\tau)\epsilon - 0.76\,\epsilon^3\sin^3\gamma\tau,$$
$$\gamma = 1.52.$$

If the axes are rotated through 45° by the substitutions

$$x = \frac{1}{\sqrt{2}}(x' - y'), \qquad y = \frac{1}{\sqrt{2}}(x' + y'),$$

the projections on the co-ordinate planes of reference are as follows:

x'y'-plane

$$\frac{\{x' - \sqrt{2}(0.465 + 0.63\epsilon^2)\}^2}{(0.14\sqrt{2}\,\epsilon^2)^2} + \frac{y'^2}{(0.05\sqrt{2}\,\epsilon^2)^2} = 1.$$

The orbit is an ellipse with the major axis parallel to the x'-axis and the centre to the right of (*b*).

x'z-plane

$$x' - \sqrt{2}(0.465 + 0.77\epsilon^2) = -0.28\sqrt{2}\,z^2.$$

The orbit is a parabola similar to Fig. 7.

y'z-plane

$$y' = -0.10\sqrt{2}\,z\sqrt{\epsilon^2 - z^2}.$$

The orbit is a double loop similar to Fig. 8. The composite result is a figure-8 curve somewhat similar to the one shown in Fig. 9.

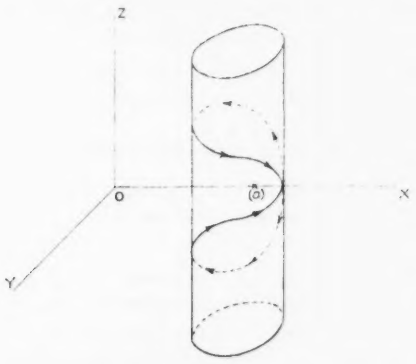

FIG. 9

The figure-8 curve is symmetrically situated with respect to the diagonal of the square of which the finite masses are the corners. The tangential

cylinder is vertical and encloses Point (*b*). The major axis of its elliptical cross-section is along the diagonal of the square. The point where the figure-8 curve crosses the diagonal lies on the side of (*b*) remote from the origin.

Libration point (c) (0.625, 0.625)

$$x = x(\tau) = 0.625 + \epsilon^2(-0.88 + 0.32\cos 2\gamma\tau + 0.18\sin 2\gamma\tau),$$
$$y = x(-\tau),$$
$$z = (\sin\gamma\tau)\epsilon - 0.28\,\epsilon^3\sin^3\gamma\tau,$$
$$\gamma = 1.04, \qquad \delta = 7.66\epsilon^2.$$

Rotating the axes through 45°, we find the projections on the planes of reference to be as follows:

x'y'-plane

$$\frac{\{x' - \sqrt{2}(0.625 - 0.88\,\epsilon^2)\}^2}{(0.32\sqrt{2}\,\epsilon^2)^2} + \frac{y'^2}{(0.18\sqrt{2}\,\epsilon^2)^2} = 1.$$

The orbit is an ellipse lying wholly between the origin and (*c*).

x'z-plane

$$x' - \sqrt{2}\,(0.625 - 0.56\epsilon^2) = -0.64\sqrt{2}\,z^2.$$

This is a parabola as in Fig. 7 but with the vertex between the origin and (*c*).

y'z-plane

$$y' = -0.36\sqrt{2}\,z\sqrt{\epsilon^2 - z^2}.$$

The orbit is a double loop as in Fig. 8.

The spatial curve is a figure-8 as in the previous cases. The tangential cylinder has its major cross-section axis along the diagonal of the square having the masses as corners. The cross-section of the tangential cylinder formed by the *xy*-plane lies between the origin and Point (*c*).

The corresponding spatial orbits near the other libration points can be readily found by a rotation of the *xy*-axes through multiples of 45°.

13. Convergence of the Solutions

Only the formal construction of the periodic and asymptotic solutions has been made. As the solutions are power series in a parameter, the question of their convergence arises.

With respect to the periodic solutions, their convergence may be established by an existence proof in which use is made of Poincaré's extension to Cauchy's theorem (22, Vol. 1, pp. 338-343) showing that periodic solutions exist and will converge for all values of the time, provided ϵ is sufficiently small numerically. Alternatively, we may use MacMillan's theorem (19) where it is shown that, if the formal construction of periodic solutions can be made, such solutions will converge for all values of the time, provided the parameter, in terms of which the solutions are expanded, is sufficiently small numerically.

The convergence of asymptotic solutions was treated by Poincaré (22, Vol. 1, pp. 338-343). He showed that such solutions will converge as t approaches $+\infty$ provided the real part of the characteristic exponent in terms of which the solutions are expanded is real and negative. This condition is satisfied in the case of the asymptotic orbits that have been constructed.

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Note:—References 1-6 deal with periodic orbits, References 7-13 with asymptotic orbits.

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